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Bit strings from $\mathcal{N} = 4$ gauge theory

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Abstract: We present an improvement of the interacting string bit theory proposed in [1], designed to reproduce the non-planar perturbative amplitudes between BMN operators in $\mathcal{N} = 4$ gauge theory. Our formalism incorporates the effect of operator mixing and all non-planar corrections to the inner product. We use supersymmetry to construct the bosonic matrix elements of the light-cone hamiltonian to all orders in g_2, and make a detailed comparison with the non-planar amplitudes obtained from gauge theory to order g_2^2. We find a precise match.

Keywords: 1/N Expansion, Penrose limit and pp-wave background.

Contents

1. Introduction

The correspondence [2] between $\mathcal{N}=4$ gauge theory at large R-charge and string theory in a plane wave geometry [3] provides a promising new example of a string/gauge theory duality, opening up a new road for studying both sides of the duality. The BMN dictionary, between single and multi-trace operators and single and multi-string states, has by now withstood many tests, but several puzzles remain. In particular, it would be desirable to have a precise characterization of the degrees of freedom within the $\mathcal{N}=4$ theory that survive the BMN limit, and a comprehensive formalism for describing their interactions. Ideally, such a formalism should allow for a clear comparison with the perturbative continuum string field theory, as well as encompass all non-perturbative D-brane configurations.

In [1], an interacting string bit model [4] was proposed with the aim of providing a useful interpolating formalism between the gauge theory and the dual continuum string theory. The idea is to design the string bit hamiltonian such that its matrix elements reproduce the perturbative gauge theory amplitudes, including the non-planar corrections. Since the appearance of [1], however, more precise insights into the structure of both the gauge theory and the bit string theory made clear that the original proposal of [1] needs refinement. Recent work by several groups [5]–[7] (following up on earlier works [8]–[12]) has produced a quite complete result for the leading non-planar contributions to operator mixing coefficients [13] and conformal dimensions for all bosonic two impurity

BMN operators. In this paper, we continue the program proposed in [1], while incorporating these new results. Using the gauge theory and supersymmetry as our guide, we construct the matrix elements of the light-cone hamiltonian to all orders in g_2. We compute the order g_2^2 matrix elements of H between bosonic two impurity states, and find a precise correspondence with the gauge theory amplitudes.

2. Free bit string theory

We start with a short review of the bit string model proposed in [1]. The bit strings consist of J string bits, where in the end one sends $J \to \infty$. To describe the motion of the string bits, we introduce J supersymmetric phase space coordinates $\{p^i_n, x^i_n, \theta^a_n, \tilde{\theta}^a_n\}$, with $n = 1, \ldots, J$, satisfying the usual canonical commutation relations

$$[\, p^i_n, x^j_m\,] = \mathrm{i}\delta^{ij}\delta_{mn}\,, \qquad \{\theta^a_n, \theta^b_m\} = \frac{1}{2}\delta^{ab}\delta_{mn}\,, \qquad \{\tilde{\theta}^a_n, \tilde{\theta}^b_m\} = \frac{1}{2}\delta^{ab}\delta_{mn}\,. \tag{2.1}$$

Since the string bits are assumed to be indistinguishable, we must divide out the symmetric group S_J, acting via permutation on the labels n.

Upon quantization, the Hilbert space splits up as a direct sum of "twisted sectors" $\mathcal{H}_\gamma$ labelled by conjugacy classes γ of the symmetric group S_J. On $\mathcal{H}_\gamma$ we can act operators $\mathcal{O}(p, x, \theta)$ that are left invariant under the action of the centralizer subgroup C_γ of γ:

$$\{\, p^i_n, x^i_n, \theta^a_n\} \;\to\; \{\, p^i_{\sigma(n)}, x^i_{\sigma(n)}, \theta^a_{\sigma(n)}\}\,, \qquad \sigma \in C_\gamma\,. \tag{2.2}$$

Since an arbitrary group element $\gamma \in S_J$ is conjugate to a permutation of the form

$$(J_1)(J_2)\ldots(J_s) \tag{2.3}$$

with (J_i) a cyclic permutation of length J_i, we thus obtain a sum over multi-string Hilbert spaces, each string with a discretized worldsheet consisting of J_ℓ bits with $\sum_\ell J_\ell = J$.

The invariance (2.2) under the stabilizer subgroup imposes the constraint $(U_\ell)^{J_\ell} = 1$ on each string, with U_ℓ the operator that translates the string bits by one unit on the ℓ-th string. The "overall" translation operator $U = \bigotimes_\ell U_\ell$ is defined to act via

$$U X_n U^{-1} = X_{\gamma(n)} \tag{2.4}$$

with $X_n = \{p^i_n, x^i_n, \theta^a_n\}$. Correspondingly, the free light-cone supersymmetry generators and hamiltonian involve "hopping terms" that depend on the choice of twisted sector γ:

$$Q_{0,1} = Q^{(0)} + \lambda\, Q^{(1)}\,, \qquad Q_{0,2} = \tilde{Q}^{(0)} - \lambda\tilde{Q}^{(1)}\,, \qquad H_0 = H^{(0)} + \lambda\, H^{(1)} + \lambda^2 H^{(2)}\,, \tag{2.5}$$

with

$$\begin{aligned}
Q^{(0)} &= \sum_n \left(p^i_n \gamma_i\, \theta_n - x^i_n(\gamma_i \Pi)\tilde{\theta}_n \right), & Q^{(1)} &= \sum_n \left(x^i_{\gamma(n)} - x^i_n \right) \gamma_i\, \theta_n \\
H^{(0)} &= \sum_n \left(\frac{1}{2}(p^2_{i,n} + x^2_{i,n}) + 2\mathrm{i}\, \tilde{\theta}_n \Pi \theta_n \right), & & \\
H^{(1)} &= -\sum_n \mathrm{i}\left(\theta_n \theta_{\gamma(n)} - \tilde{\theta}_n \tilde{\theta}_{\gamma(n)} \right), & H^{(2)} &= \sum_n \frac{1}{2}\left(x^i_{\gamma(n)} - x^i_n \right)^2 .
\end{aligned} \tag{2.6}$$

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These expressions are the most straightforward discretization of the supercharges of the continuum string theory given in [14]. It is easy to verify that they generate a closed supersymmetry algebra. Note further that the hopping terms are defined such that they just act within each separate string.

To proceed, it will be useful to introduce creation and annihilation operators for the individual string bits via

$$\begin{aligned} x^i_m &= \frac{1}{\sqrt{2}}(a^i_m + a^{i\,\dagger}_m) & p^i_m &= \frac{\mathrm{i}}{\sqrt{2}}(a^i_m - a^{i\,\dagger}_m) \\ \theta_n &= \frac{1}{2}\left(\beta_n + \beta_n{}^\dagger\right) & \Pi\theta_n &= \frac{i}{2}\left(\tilde{\beta}_n - \tilde{\beta}_n{}^\dagger\right) \\ \tilde{\theta}_n &= \frac{1}{2}\left(\tilde{\beta}_n + \tilde{\beta}^\dagger_n\right) & \tilde{\theta}_n\Pi &= \frac{i}{2}\left(\beta_n - \beta^\dagger_n\right). \end{aligned} \tag{2.7}$$

This choice of fermionic creation and annihilation operators breaks the SO(8) rotation symmetry down to SO(4) $\times$ SO(4).

We can now define the untwisted vacuum state $|0\rangle$ as the simple tensor product of the vacuum states of the individual string bits

$$a^i_m|0\rangle = 0\,, \qquad \beta^{\dot a}_m|0\rangle = 0\,, \qquad \tilde{\beta}^{\dot a}_m|0\rangle = 0 \qquad \langle 0|0\rangle = 1\,. \tag{2.8}$$

This state describes J disconnected string bits, i.e. J separate short strings of unit length, all in their ground state. We wish define operators that, when acting on $|0\rangle$ produce the ground state of a collection of long strings of length J_ℓ with $\sum_\ell J_\ell = J$.

For all elements $\sigma \in S_J$, we associate a corresponding twist operator Σ_σ that implements the permutation σ on the string bits:

$$\Sigma_\sigma X_n = X_{\sigma(n)}\,\Sigma_\sigma\,. \tag{2.9}$$

This operator acts non-trivially on the twisted sectors via

$$\Sigma_\sigma : \quad \mathcal{H}_\gamma \to \mathcal{H}_{\gamma\circ\sigma}\,. \tag{2.10}$$

In particular, we can define the twisted vacuum states $|\gamma\rangle \in \mathcal{H}_\gamma$ as follows [15]

$$|\gamma\rangle = \frac{1}{\sqrt{N_\gamma}}\sum_{\sigma\in\gamma}\Sigma_\sigma\,|0\rangle\,. \tag{2.11}$$

Here N_γ is the number of elements in the conjugacy class γ. This twisted vacuum state also satisfies

$$a^i_m\,|\gamma\rangle = 0\,, \qquad \beta^{\dot a}_m\,|\gamma\rangle = 0\,, \qquad \tilde{\beta}^{\dot a}_m\,|\gamma\rangle = 0\,, \qquad \langle\gamma|\gamma\rangle = 1\,. \tag{2.12}$$

This ground state $|\gamma\rangle$ describes a collection of strings in their ground state, of length J_i depending on the decomposition (2.3) of γ into cyclic permutations. In particular, the single string state corresponds to the long cycle $\gamma_1 = (1\;2\ldots J)$, and double string states to permutations of the form $\gamma_2 = (1\;2\ldots J_1)(J_1\ldots J)$.

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3. BMN dictionary

BMN proposed a concrete dictionary between operators in the string bit theory and operators of large R charge in $\mathcal{N}=4$ gauge theory. This correspondence is based on the identification of the string light-cone hamiltonian with $H = \Delta - J$, where J (the total number of string bits) equals the total R charge. The hopping parameter λ in the string bit hamiltonian gets identified with the 't Hooft coupling of the gauge theory

$$\lambda^2 = \frac{g_{YM}^2 N}{8\pi^2} \,. \tag{3.1}$$

The free string bit theory then describes the planar limit $N \to \infty$. Our goal is to construct the effective string interactions that arise for $g_2 = J^2/N$ finite.

In [5]–[7] a complete analysis was given of the leading order non-planar corrections to the conformal dimension of a specific class of BMN operators. We will test our formalism by showing that it reproduces the same results. The normalized one-string BMN operators considered are of the form

$$O_p^J = \frac{1}{\sqrt{JN^{J+2}}} \sum_{l=1}^{J} e^{2\pi\, ipl/J} \mathrm{Tr}(\phi\, Z^l \psi\, Z^{J-l}) \,. \tag{3.2}$$

To write the corresponding bit string state we introduce the operator [15]

$$O_{p,\gamma_1}^J = \frac{1}{J} \left(\sum_{k=1}^{J} a^\dagger_{\gamma_1(k)} e^{-2\pi\, ipk/J} \right) \left(\sum_{l=1}^{J} b^\dagger_{\gamma_1(l)} e^{2\pi\, ipl/J} \right) \tag{3.3}$$

where γ_1 is the long cycle of length J. The associated state is obtained by acting on the single string vacuum state $|\gamma_1\rangle$, and then summing over all conjugations of γ_1:

$$|O_p^J\rangle = \frac{1}{\sqrt{J!J}} \sum_{\tilde{\gamma}_1 = h^{-1}\gamma_1 h} O^J_{p,\tilde{\gamma}_1} |\tilde{\gamma}_1\rangle \,, \tag{3.4}$$

with $h \in S_J$. This state is normalized to have unit norm.

Similarly, we can construct a normalized two-string state corresponding to the normalized double trace BMN operator

$$\frac{1}{\sqrt{J_1(J-J_1)N^{J+2}}} \sum_{l=1}^{J_1} e^{2\pi\, ikl/J_1}\, \mathrm{Tr}(\phi\, Z^l \psi\, Z^{J_1-l})\, \mathrm{Tr}(Z^{J-J_1}) \tag{3.5}$$

via

$$|O_k^{J_1}\rangle = \frac{1}{\sqrt{J!J_1(J-J_1)}} \sum_{\tilde{\gamma}_2 = h^{-1}\gamma_2 h} O^{J_1}_{k,\tilde{\gamma}_2} |\tilde{\gamma}_2\rangle \tag{3.6}$$

with

$$O_{k,\gamma_2}^{J_1} = \frac{1}{J_1} \left(\sum_{l=1}^{J_1} a^\dagger_{\gamma_2(l)} e^{-2\pi\, ikl/J_1} \right) \left(\sum_{l'=1}^{J_1} b^\dagger_{\gamma_2(l')} e^{2\pi\, ikl'/J_1} \right) \tag{3.7}$$

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where γ_2 is decomposed as $(1\ 2 \dots J_1)(J_1+1 \dots J)$. Finally, the other type of two-string state corresponding to

$$\frac{1}{N^{J+2}} \operatorname{Tr}(\phi\, Z^{J_1})\ \operatorname{Tr}(\psi\, Z^{J-J_1}) \tag{3.8}$$

is

$$|O_0^{J_1\, J_2}\rangle = \frac{1}{\sqrt{J! J_1\,(J-J_1)}} \sum_{\tilde{\gamma}_2 = h^{-1}\gamma_2 h} O_{0,\tilde{\gamma}_2}^{J_1 J_2} |\tilde{\gamma}_2\rangle \tag{3.9}$$

with

$$O_{0,\gamma_2}^{J_1 J_2} = \frac{1}{\sqrt{J_1\,(J-J_1)}} \left(\sum_{l=1}^{J_1} a^\dagger_{\gamma_2(l)}\right) \left(\sum_{l'=J_1+1}^{J} b^\dagger_{\gamma_2(l')}\right). \tag{3.10}$$

In the following we will often use the notation

$$|1,p\rangle = |O_p^J\rangle\,, \qquad |2,k,y\rangle = |O_k^{J_1}\rangle\,, \qquad |2,y\rangle = |O_0^{J_1\, J_2}\rangle\,, \tag{3.11}$$

with $y = J_1/J$ a sub-unitary parameter that parametrizes the relative length of the two strings. These three states are all eigenstates of the free hamiltonian H_0 with respective eigenvalues equal to $E_p = 2 + \lambda'\, p^2$, $E_k = 2 + \lambda' k^2/y^2$ and 2, with

$$\lambda' = \frac{8\pi^2 \lambda^2}{J^2}\,. \tag{3.12}$$

Two final comments: (1) Note that the definition (2.8) of the bit string vacuum state is uniquely selected by requiring that is should correspond to the BPS operator $\operatorname{Tr}(Z^J)$: both are the lowest energy eigen states. (2) Above we have made a direct correspondence between the single and double trace BMN operators and one and two string states. As we will see shortly, this identification is in fact somewhat premature, since upon turning on the effective string coupling

$$g_2 = \frac{J^2}{N} \tag{3.13}$$

single and multiple trace operators will inevitably start to mix. For now, however, we will adopt the above direct identification between the BMN operators and string bit states, leaving the discussion of possible redefinitions to the concluding section. First we wish to determine the form of the bit string interactions, using the gauge theory as our guide.

4. Inner product at finite g_2

A characteristic aspect of the gauge theory is that, even at zero 't Hooft coupling λ, the overlap between the BMN operators has a non-trivial expansion in terms of $g_2 = J^2/N$, because free Wick contractions can still generate a sum over non-planar diagrams. In particular, there is a non-vanishing overlap between single trace and multi-trace BMN operators [9, 8, 11].

Since the $\lambda = 0$ theory is free, this structure can be explicitly worked out by keeping track of the permutations $\sigma \in S_J$ encoded in the Wick contractions between the J string bits of the "in" and "out" operators [8]. Since our goal is to construct the bit string model

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in such a way that it reproduces the gauge theory amplitudes, we need to incorporate this structure by means of an appropriate choice of inner product. Luckily, permutations of the string bits are already a natural part of the story.

Recall that any permutation σ can be factorized into a product of simple permutations of the form (nm). Let $h(\sigma)$ be the minimal number of simple permutations needed in this factorization of σ. The inner product, that realizes the combinatorics of the free gauge theory amplitudes in the string bit language, is of the form

$$\langle \psi_1 | \psi_2 \rangle_{g_2} = \langle \psi_1 | S | \psi_2 \rangle_0 , \tag{4.1}$$

where S is the following weighted sum over all possible permutation operators

$$S = \sum_{\sigma} N^{-2h(\sigma)} \, \Sigma_\sigma . \tag{4.2}$$

To understand the structure of this inner product, let us consider the first few terms in the expansion (4.2) a bit more closely. Writing

$$S = 1 + \frac{1}{N}\Sigma_2 + \frac{1}{N^2}\,\Sigma_3 + \cdots \tag{4.3}$$

we find for the first order term

$$\Sigma_2 = \sum_{n<m} \Sigma_{(nm)} , \tag{4.4}$$

with (nm) the simple permutation of order 2. This operator Σ_2 represents a basic cubic string joining and splitting interaction. The special role of Σ_2 will become more apparent in the following.

The second order term Σ_3 is

$$\Sigma_3 = \sum_{\substack{m<n \\ m<k<l \\ k,l\neq n}} \Sigma_{(mn)(kl)} + \sum_{m<n<k} \big(\Sigma_{(mnk)} + \Sigma_{(knm)}\big) . \tag{4.5}$$

When acting on a single string state, it can either split the string into three separate strings, or induce a subsequent splitting and joining, producing a new reordered single string state. It is straightforward to verify that the Feynman diagram produced by the corresponding free Wick contraction between single trace “in” and “out” BMN operators has genus 1. Now, since $(mnk) = (mk)(kn)$, we see that

$$\Sigma_3 = \frac{1}{2}\left((\Sigma_2)^2 - J(J-1)\right) . \tag{4.6}$$

The c-number term becomes negligible in the limit of large J, since Σ_3 in (4.3) comes with a prefactor of $1/N^2$. The physical meaning of the identity (4.6) is that all second order string interactions can be thought of as the result of two elementary string interactions. Generalizing this observation to higher orders, it is natural to suspect that in the the limit of large J, we can write

$$S = e^{g_2 \Sigma} , \qquad \Sigma \equiv \frac{1}{J^2}\,\Sigma_2 . \tag{4.7}$$

We claim that the inner product (4.1) with (4.7) indeed corresponds to that of the free gauge theory at large J but finite g_2.

As a specific check, let us compute the genus h contribution to the overlap between two single string vacuum states

$$\frac{1}{(2h)!}\,\langle\gamma_1|(\Sigma_2)^{2h}|\gamma_1\rangle\,. \tag{4.8}$$

This contribution is equal to the total number of products of $2h$ simple permutations that, when acting on a long cycle (single string) produce another long cycle. This number can be evaluated as follows (see e.g. the discussion in [8]): The product Σ_2^{2h} involves a sum over $2h$ pairs of positions, which (absorbing the factor $1/(2h)!$) can be assumed to be ordered. The $2h$ bit pairs split up the J bits into $4h$ groups. Placing the J string bits along a circle, this produces a $4h$-gon, on which the $2h$ pairs represent a specific gluing rule: each two corner points of the $4h$-gon connected by a simple permutation must be glued together. This gluing rule reflects the correspondence between simple permutations and elementary string splitting or joining interactions. The condition that the product of simple permutations maps a long cycle to another long cycle, now translates into the condition that the gluing produces a surface of genus h. Via this reasoning, one obtains that (4.8) equals the number of ways of dividing J bits into $4h$ groups (which for large J equals $J^{4h}/4h!$) times the number of ways of gluing a $4h$-gon into a genus h surface (which is known to be equal to $\frac{(4h-1)!!}{2h+1}$). So our bit string inner product indeed reproduces the gauge theory result

$$\langle\gamma_1|S\,|\gamma_1\rangle = \sum_{h=0}^{\infty}\frac{1}{(2h+1)!}\left(\frac{g_2}{2}\right)^{2h} = \frac{2}{g_2}\sinh\left(\frac{g_2}{2}\right). \tag{4.9}$$

As a further concrete check on the above reasoning, we have explicitly worked out the genus 1 and 2 contributions in appendix B.

Another check on our inner product is obtained by considering the matrix elements of the first and second order terms in S between the special class of states introduced earlier. A straightforward calculation (along the lines of [15]) shows that a single action of Σ produces the same non-zero "three point functions" between the normalized two-impurity states as those obtained in the free gauge theory [9, 11, 8]

$$\begin{aligned} C_{pky} \equiv \langle\, 2,k,y|\,\Sigma\,|\,1,p\,\rangle \;&=\; \sqrt{\frac{1-y}{Jy}}\,\frac{\sin^2(\pi p y)}{\pi^2(p-k/y)^2}\,, \\ C_{py} = \langle 2,y\,|\,\Sigma\,|\,1,p\,\rangle \;&=\; -\frac{\sin^2(\pi p y)}{\sqrt{J}\pi^2 p^2}\,. \end{aligned} \tag{4.10}$$

From the definition of Σ, it is furthermore clear that these "three point functions" form a complete set in the sense that

$$\Sigma\,|1,p\rangle = \sum_{k,y} C_{pky}\,|2,k,y\rangle \;+\; \sum_{y} C_{py}|2,y\rangle\,. \tag{4.11}$$

As a technical aside, we note that the above decomposition relation in fact reveals a conceptual subtlety, which was noted in [15]. Namely, the sum over k in (4.11), strictly speaking, must be extended to include values of order J_1. In this regime, however, one can

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no longer make the approximation $\sin^2(\pi(py-k)/J_1) \simeq \pi^2(py-k)^2/J_1^2$ that was used to derive (4.10). Problems of this sort often arise in discretized models, and we will deal with it in the usual manner: we will simply truncate the Hilbert space of the bit string model to include only those frequencies k negligibly small compared to J. This restriction should become insignificant upon taking the large J limit.

The second order matrix element of S between the single string states, representing the one-loop contribution due to successive splitting and joining, can be similarly be obtained, either via direct computation, or by using factorization

$$A_{pq} \equiv \frac{1}{2}\langle\, 1, q|\Sigma^2|\, 1, p\,\rangle \;=\; \frac{1}{2}\left(\sum_{k,y} C_{pky}C_{qky} + \sum_{y} C_{py}C_{qy}\right). \tag{4.12}$$

This relation matches the factorization property of the inner product of the free gauge theory, which was first derived in [12]. The explicit form of A_{pq} is as given in [9, 8].

Finally, we need to emphasize that the above amplitudes do not yet represent proper string interactions. String interactions are associated with non-trivial matrix elements of the light-cone hamiltonian. The g_2-dependence of the inner product can obviously be transformed away by a redefinition of the single and multi-string states (see the concluding section). For now we will stick to the above basis, so that the relation to the gauge theory is most apparent.

5. Interactions and supersymmetry

The modification (4.1) of the inner product at finite string coupling g_2 indicates that we must also add new interaction terms to the supersymmetry generators and hamiltonian. Matrix elements of the hamiltonian at non-zero g_2 can be expressed in terms of the bare inner product (the one at $g_2=0$) via

$$\langle\, \psi_2|H\,|\,\psi_1\rangle_{g_2} \;=\; \langle\, \psi_2|SH\,|\,\psi_1\rangle_0\,. \tag{5.1}$$

Hermiticity of H thus requires that

$$H = H^\dagger = S^{-1}H^{\dagger 0}S\,, \tag{5.2}$$

where $H^{\dagger 0}$ denotes the hermitean conjugate relative to the bare inner product. A similar condition holds for the supersymmetry generators.

Another non-trivial consistency requirement is the closure of the interacting light-cone supersymmetry algebra (here $I, J = 1, 2$ — see eq. (2.5))

$$\delta^{IJ}\{Q_I^{\dot a}, Q_J^{\dot b}\} = \;\delta^{\dot a\dot b}H \,+\, J^{\dot a\dot b}\,, \tag{5.3}$$

where $J^{\dot a\dot b}$ is a suitable contraction of gamma matrices with the SO(4) $\times$ SO(4) Lorentz generators J^{ij}, see [14].

In this section we will write a new Ansatz for $Q^{\dot a}$ and H, that will be hermitean relative to the new inner product, and will produce non-trivial string interactions proportional to g_2. Our Ansatz will generate the light-cone supersymmetry algebra (5.3), but only at the linearized level in the fermions, that is, when inserted between string states with only bosonic excitations (or between a purely bosonic and a fermionic one). In principle, it should be straightforward to correct our Ansatz for Q by means of non-linear fermionic terms, so that the algebra closes for all fermionic states as well. Our main interest in the following, however, will be to compare our model with the gauge theory computations, which so far have been done for bosonic states. Because we expect that the non-linear fermionic correction terms in the end will not modify these bosonic amplitudes, we will leave their study to a future work.

To write the interacting generators, we will use the correspondence with the gauge theory as our guide. The basic idea will be the following. We will assume that the free supersymmetry generators can be split into two terms

$$Q_0 = Q_0^{>} + Q_0^{<} \,, \tag{5.4}$$

such that, in the interacting theory, $Q_0^{<}$ will receive correction terms that induce string splitting and joining *only* when acting on states to the right, while $Q_0^{>}$ will induce string splitting and joining *only* when acting on states to the left. The underlying motivation for this assumption is that, in the correspondence with the gauge theory, $Q_0^{<}$ represents an interaction term of the form $\mathrm{Tr}(\theta[\bar{Z},\bar{\phi}])$ which naturally acts via a *double* Wick contraction on the "in" BMN state, while only with a *single* contraction on the "out" state. The double Wick contraction can split up a trace, or join a product of two traces into a single trace, while a single contraction can not.

Let us give an explicit example. The Wick contraction between

$$Q^{<} = \mathrm{Tr}\left(\theta[\overline{Z},\overline{\phi}]\right) \qquad \text{and} \qquad O = \sum_{l=0}^{J} q^l\, \mathrm{Tr}\left(\phi Z^l \psi Z^{J-l}\right) \tag{5.5}$$

with $q = e^{2\pi i p/J}$ has been found to be equal to [8]

$$Q^{<} O \,=\, -iN\,(q-1) \sum_{l=0}^{J-1} q^l \left(\theta Z^l \psi Z^{J-l-1}\right) - \tag{5.6}$$

$$-\, i \frac{q}{q-1} \sum_{J_1=1}^{J-1} \left(\theta Z^{J_1}\right)\left(Z^{J-J_1-1}\psi\right)\left(1+q^{-1}-q^{J_1}-q^{-J_1-1}\right) - \tag{5.7}$$

$$-\, i \sum_{J_1=1}^{J-1} \sum_{l=J_1+1}^{J} q^l \left(1-q^{-J_1-1}\right)\left(Z^m\right)\left(\theta Z^{l-J_1-1}\psi Z^{J-l}\right). \tag{5.8}$$

In string bit language, the single trace term corresponds to the free action of the supercharge, while the double trace terms are due to an interaction term in $Q^{<}$ proportional to g_2, that induces a single string splitting. In contrast, the Wick contraction between

$$Q^{>} = \mathrm{Tr}\left(\overline{\theta}[Z,\phi]\right) \qquad \text{and} \qquad O' = \sum_{l=0}^{J} q^l\, \mathrm{Tr}\left(\theta Z^l \psi Z^{J-l}\right) \tag{5.9}$$

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is simply

$$Q^{>}O' = -iN\sum_{l=0}^{J-1} q^l \left([Z,\phi]Z^l\psi Z^{J-l-1}\right). \tag{5.10}$$

Hence this term $Q^{>}$, which is the hermitean conjugate of $Q^{<}$, acts just like the free supercharge.

We wish to incorporate this same structure into the definition of the supersymmetry generators of the bit string theory. The above two gauge theory calculations suggest that the division (5.4) should be made such that terms of the form $\beta^\dagger_m a_n$ are part of $Q_0^{<}$ and will receive interaction terms proportional to g_2 when acting to the right, while all terms of the form $\beta_m a^\dagger_n$ are part of $Q_0^{>}$ and remain free when acting to the right.

With this motivation, we will now adopt the following Ansatz for the supersymmetry generators for finite g_2

$$Q = Q_0^{>} + S^{-1}Q_0^{<}S\,, \tag{5.11}$$

where the $>$ superscript indicates the terms that contain fermionic annihilation operators β_m only, while $<$ denotes terms with only $\beta^\dagger_m$'s. In particular,

$$Q^{(1)>} = -\sum_n \left(x^i_{\gamma(n)} - x^i_n\right)\gamma^i\beta_n\,; \qquad Q^{(1)<} = -\sum_n \left(x^i_{\gamma(n)} - x^i_n\right)\gamma^i{\beta_n}^\dagger\,. \tag{5.12}$$

The Ansatz (5.11) by construction satisfies the hermiticity condition $Q^\dagger = Q$, relative to the new inner product.

A priori, since S contains terms of arbitrarily high powers in g_2, the new supersymmetry generators Q in (5.11) appear to have an infinite g_2 expansion. The gauge theory supercharges, on the other hand, can effectuate (if we assume they need at least one Wick contraction with either the "in" our "out" BMN state) at most a single string splitting or joining interaction. It indeed turns out that, also in our bit string model, only a linear interaction term survives

$$Q = Q_0 + g_2\left[Q_0^{<},\Sigma\right], \tag{5.13}$$

provided we take the strict large J limit. This simplification of the Ansatz (5.11) follows from the fact that in this limit

$$[[\,Q_0^{<},\Sigma\,],\Sigma\,] = 0\,. \tag{5.14}$$

To derive this identity, we note that the double commutator with Σ can be reduced to a triple (rather than quadruple) summation over the J sites, since the indices in the simple permutations in the two Σ factors have to coincide, or differ by at most one unit, in order to give a non-zero result (for finite J). This triple summation is insufficient to overcome the $1/J^4$ pre-factor, and thus the double commutator (5.14) vanishes in the strict large J limit.

It is straightforward to obtain an explicit form of the interaction term in (5.13), by letting it act on an arbitrary state $|\psi_{\gamma_1}\rangle$ in a twisted sector $\mathcal{H}_\gamma$:

$$[\,Q_0^{<},\Sigma\,]|\psi_{\gamma_1}\rangle = \frac{1}{J^2}\sum_{m<n}[\,Q_0^{<},\Sigma_{mn}\,]\,|\psi_{\gamma_1}\rangle \;=\; \frac{1}{J^2}\sum_{m<n}\Sigma_{mn}\left(Q^{<}_{\gamma_2} - Q^{<}_{\gamma_1}\right)|\psi_{\gamma_1}\rangle\,, \tag{5.15}$$

where $\gamma_2 = \gamma_1 \circ (mn)$. Inserting the explicit form (5.12) of the supersymmetry generator gives

$$[\,Q_0^<\,,\Sigma\,] \;=\; \frac{\lambda}{2J^2}\sum_{m,n}\Sigma_{mn}\Big((x^i_{\gamma(m)}-x^i_{\gamma(n)})\gamma_i\,\beta^\dagger_m + (x^i_m-x^i_n)\gamma_i\,\beta^\dagger_{\gamma^{-1}(m)} + \\ + \delta_{n\gamma(m)}\Big(x^i_n\gamma_i\beta^\dagger_m - x^i_m\gamma_i\beta_n{}^\dagger\Big)\Big)\,. \tag{5.16}$$

This interaction term has a finite strength, because it is involves a double sum over J sites, which compensates for the $1/J^2$ factor in front.

Let us now verify that the above interaction term, when acting on the two impurity single string state $|O^J_p\rangle$, produces the same result (5.6) as found the gauge theory. We can choose the one string state $|O^J_p\rangle$ to lie in the twisted sector labelled by $\gamma_1 = (1\;2\ldots J)$. Acting with the interaction term of the supercharge then produces[1]

$$\frac{1}{J^2}\sum_{m<n}\Sigma_{mn}\,(\,Q^<_{\gamma_2} - Q^<_{\gamma_1})|O^J_p\rangle = \frac{\lambda}{J}\sum_{J_1}\Sigma_{JJ_1}\Big[(a^i_J - a^i_{J_1})\gamma^i(\beta^\dagger_{J_1-1} - \beta^\dagger_{J-1})\Big]\sum_{k,l=1}^{J} a^\dagger_k b^\dagger_l q^{l-k}\,|\gamma_1\rangle$$

with $q = e^{-\frac{2\pi i p}{J}}$. Decomposing the sum over the position of the impurity $b^\dagger_l$ as

$$\sum_{l=0}^{J_1-1} b^\dagger_l q^l \equiv A \qquad \sum_{l=J_1}^{J-1} b^\dagger_l q^l \equiv B \tag{5.17}$$

the action of $[Q^<\,,\Sigma_{J_1J}]$ on the one-string state is given by

$$[Q^<\,,\Sigma_{J_1J}]|O^J_p\rangle = \Big(1-q^{-J_1}\Big)\Big[\Big(\beta^\dagger_{J_1-1}B - \beta^\dagger_{J-1}A\Big) + \Big(-\beta^\dagger_{J-1}B + \beta^\dagger_{J_1-1}A\Big)\Big]\;|\gamma_2\rangle\,, \tag{5.18}$$

where the first two terms on the right-hand side of (5.18) describe states with a fermionic impurity on a string of length J_1 respectively $J{-}J_1$ and a bosonic impurity on the complementary string, while the last two terms in (5.18) are states with both impurities sitting on the same string of length J_1 and respectively $J-J_1$. Given the fact that one is supposed to sum over all cyclic permutations, the states where the bosonic impurity is placed on a different string than the fermionic one[2] yield

$$\Big(1-q^{-J_1}\Big)\left(\beta^\dagger_{J_1-1}b^\dagger_{J-1}\sum_{l=J_1}^{J-1} q^l\right)|\gamma_1\rangle \;=\; \frac{1}{q-1}\Big(2-q^{-J_1}-q^{J_1}\Big)\beta^\dagger_{J_1-1}b^\dagger_{J-1}|\,\gamma_2\,\rangle \tag{5.19}$$

which by the dictionary we have established between the string bit and the gauge theory is equal to

$$\frac{1}{q-1}\Big(2-q^{-J_1}-q^{J_1}\Big)\,\mathrm{Tr}\left(Z^{J_1-1}\theta\right)\mathrm{Tr}\left(Z^{J-J_1-1}\Psi\right). \tag{5.20}$$

[1] Here, in order to compare with the gauge theory calculation leading to (5.6), we keep in Q only the term with a^i annihilators, leaving out the creation and b^i modes. The b^i terms would correspond in the gauge theory to a term of the form $\mathrm{Tr}\,\theta[\bar\psi,\bar Z]$.

[2] Just like in the gauge theory, a one-string state of length J and momentum p with a single impurity vanishes for non-zero p, due to the imposed invariance under cyclic permutations of γ_1.

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The states where the impurities sit on the same string are

$$\left(1-q^{-J_1}\right)\beta^\dagger_{J-1}\sum_{l=J_1}^{J-1} b^\dagger_l q^l|\,\gamma_2\,\rangle \tag{5.21}$$

and the same dictionary relates them to a gauge theory operator

$$\left(1-q^{-J_1}\right)\sum_{l=J_1}^{J-1} q^l \;\mathrm{Tr}\left(Z^{J-1}\right)\;\mathrm{Tr}\left(Z^{l-J_1}\Psi Z^{J-2-l}\theta\right)\,. \tag{5.22}$$

Finally, note that out the four possible configurations of (5.18) we have discussed only two, with the bosonic impurity placed on the string of length $J-J_1$. The other two are taken into account as we sum over J_1: they appear when J_1 equals $J_2 = J-J_1$ with the bosonic impurity placed on the string of length $J-J_2 = J_1$. Thus in summing over J_1 we find that $[Q^<, \Sigma_{J_1 J}]$ is given by twice (5.19) and (5.21). Comparing the string bit calculation with the gauge theory we see that (5.22) reproduces the last term in (5.6) while (5.20) corresponds to the second term in (5.6). There are subtle differences, which however disappear in the large J limit, due to the fact that the action of $\mathrm{Tr}(\theta[Z,\phi])$ in the gauge theory doesn't conserve R-charge: it reduces or increases the number of string bits J by one unit. The supersymmetry generators in the string bit picture, on the other hand, preserve the total length of the bit strings.

6. Matrix elements of the hamiltonian

In this section we will evaluate the matrix elements of the string bit hamiltonian, and compare it with the ones computed in gauge theory. So far, all gauge theory computations have been done for BMN states with two bosonic impurities.

Starting from our Ansatz (5.11) for the supercharges, we can now define the matrix elements of H via

$$\langle\psi_2|(\delta^{\dot a\dot b}H + J^{\dot a\dot b})|\psi_1\rangle_{g_2} = \delta^{IJ}\langle\psi_2|\,S\,\{Q^{\dot a}_I, Q^{\dot b}_J\}|\psi_1\rangle_0\,. \tag{6.1}$$

First, however, we need to verify whether the supersymmetry algebra is indeed satisfied.

Let us first look at the interaction terms linear in g_2. Since the SO(4) × §0(4) rotation symmetry is purely kinematical, it is clear that the rotation generators J_{ij} should not receive any g_2-corrections. Consistency of the algebra to linear order in g_2 thus requires that

$$\delta^{IJ}\{(Q_0)^{\dot a}_I, [(Q^<_0)^{\dot b}_J, , \Sigma]\} \;=\; \delta^{\dot a\dot b}\,V_1 \tag{6.2}$$

with V_1 the order g_2 interaction term of the hamiltonian.[3] We will now show that the above equation is indeed satisfied at the linearized level in the fermions.

[3] Note however that, due to the g_2 dependence of the inner product, the first order g_2-correction in the matrix element of H in fact has two contributions:

$$\langle\psi_2|H_1|\psi_1\rangle = \langle\psi_2|(\Sigma H_0 + V_1)|\psi_1\rangle\,.$$

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Explicit evaluation of the anti-commutator on the left-hand side of (6.2) gives (here we are omitting the spinor indices)

$$\{Q_0,[Q_0^<,\Sigma]\} = \frac{1}{2J^2}\sum_{m<n}\Sigma_{mn}\left(\{Q_{\gamma_1}+Q_{\gamma_2},Q^<_{\gamma_2}-Q^<_{\gamma_1}\}-[Q_{\gamma_1}-Q_{\gamma_2},Q^<_{\gamma_2}-Q^<_{\gamma_1}]\right), \qquad (6.3)$$

where as before, we imagine that the operator is acting on a state in the twisted sector γ_1; the permutation γ_2 is obtained from γ_1 by applying the transposition (mn).

The first term on the right-hand side is an anti-commutator of two expressions of the same general form is in (5.16). It is easy to see that this anti-commutator will produce an expression of the form $\delta^{\dot a\dot b}V_1$. The second term, on the other hand, is a *commutator* between anti-commuting quantities, and is not proportional to $\delta^{\dot a\dot b}$. However, it is necessarily quadratic in the fermion oscillators; moreover, one can show it contains both fermionic annihilation and creation operators (each acting at different locations), and therefore gives a vanishing contribution when acting on purely bosonic states in either direction. As stated before, we expect that this term can be cancelled by adding higher order fermionic terms to Q, without modifying the bosonic part of H.

It is not difficult to show that, when evaluated between bosonic states, the supersymmetry algebra also closes to second order in g_2. Because the supercharges themselves are linear in g_2, this is sufficient. Hence we can use (6.1) to define the bosonic matrix elements of H to all orders in g_2. Inserting the original Ansatz (5.11) for the supercharges, we obtain

$$\langle\psi_2|(\delta^{\dot a\dot b}H+J^{\dot a\dot b})|\psi_1\rangle_{g_2} = \delta^{IJ}\langle\psi_2|\,S\,(Q_0^>)^{\dot a}_I S^{-1}(Q_0^<)^{\dot b}_J S|\,\psi_1\rangle_0 + (\dot a\leftrightarrow\dot b)\,. \qquad (6.4)$$

Here we used that $Q_0^>$ annihilates bosonic states; we will continue to use this fact in the following. We will now explicitly evaluate these matrix elements between the class of states discussed earlier; we will work to leading order in λ' and second order in g_2.

Operator mixing at order g_2. To linear order in g_2, the hamiltonian has non-zero matrix elements between single and double string states. In the gauge theory, this corresponds to an operator mixing term between single and double trace operators. We will find an exact match with the gauge theory computations done recently in [5, 6].

Expanding (6.4) to linear order in g_2 gives

$$\langle\psi_2|H_1|\psi_1\rangle = \langle\psi_2|(H_0\Sigma+\Sigma H_0)|\psi_1\rangle\langle\psi_2|Q_0^>\Sigma\, Q_0^<|\psi_1\rangle,. \qquad (6.5)$$

It is straightforward to evaluate this amplitude between the states $|1,p\rangle$ and $|2,k,y\rangle$ introduced in section 3. We obtain

$$\langle 2,k,y|(H_0\Sigma+\Sigma H_0)|1,p\rangle = \lambda'\left(p^2+\frac{k^2}{y^2}\right)C_{pky}\,, \qquad (6.6)$$

$$\langle 2,k,y|Q_0^>\,\Sigma\,Q_0^<|1,p\rangle = \lambda'\left(\frac{pk}{y}\right)C_{pky}\,, \qquad (6.7)$$

where C_{pky} is the bare "three point function" (the matrix element of Σ) given in (4.10). The first result (6.6) is immediate. The second result (6.7) can be understood intuitively, by noting that after contracting the fermionic oscillators in $Q_0^<$ and $Q_0^>$, one is left with

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a product of the form $\partial x^i \Sigma \partial x^i$, which when evaluated gives a contribution proportional to pky times the bare three point function. We will explicitly verify this intuition in appendix A.

In a similar way, we can also evaluate the operator between $|1,p,y\rangle$ and the other two string state $|2,y\rangle$. The final result for both amplitudes

$$\begin{aligned} \langle 2,k,y|\, H_1\, |1,p\rangle &= \lambda' \left(p^2 + \frac{k^2}{y^2} - p\,\frac{k}{y}\right) C_{p\,ky} \\ \langle 2,y|\, H_1\, |1,p\rangle &= \lambda'\, p^2 C_{p\,y} \end{aligned}$$

reproduces the gauge theory result [5, 6].

In itself, this match is not yet a conclusive indication that our bit string theory reproduces the gauge theory amplitudes, because the value of mixing matrix elements depends on a choice of basis. It is an encouraging sign, however, that our present choice of basis indeed matches with that of the gauge theory.[4] A more conclusive verification of the correspondence requires going to second order in g_2.

Order g_2^2 matrix element. To compute the order g_2^2 matrix element of H between two single string states, we first expand (6.4) to second order. We find

$$\langle\, 1,q\,|\, H_2 |\, 1,p\,\rangle = X - Y + Z \tag{6.8}$$

with

$$\begin{aligned} X &= \frac{1}{2}\langle 1,q|(H_0\Sigma^2 + \Sigma^2\, H_0)|1,p\,\rangle \\ Y &= \frac{1}{2}\,\langle 1,q|\, Q_0^{>}\, \Sigma^2\, Q_0^{<}\, |1,p\,\rangle \\ Z &= \langle 1,q|\,[\, Q_0^{>}, \Sigma][\,\Sigma, Q_0^{<}\,]\,|1,p\,\rangle\,. \end{aligned}$$

The first two contributions X and Y have a similar structure as the order g_2 terms, except with Σ replaced by Σ^2. Again, X is easily evaluated, while Y can be obtained via a similar calculation as before, with a similar result

$$X = \lambda'(p^2+q^2)A_{p\,q}\,, \qquad Y = \lambda' p\, q A_{p\,q}\,. \tag{6.9}$$

Here $A_{p\,q}$ is the matrix element of the splitting-and-joining interaction term Σ^2, given in (4.12).

Finally we need to evaluate Z. This is done as follows:

$$\begin{aligned} Z &= \sum_i \Big(\langle 1,q|\,\Sigma|\,i\rangle\langle\, i\,|[Q_0^{>},\Sigma]Q_0^{<}\,|1,p\rangle \;-\; \langle 1,q|[Q_0^{>},\Sigma]Q_0^{<}\,|\,i\,\rangle\langle\, i\,|\Sigma|1,p\rangle\Big) \\ &= \sum_{k,y}\left(p\left(\frac{k}{y}-p\right) - \left(\frac{k}{y}\right)\left(q-\frac{k}{y}\right)\right) C_{p\,ky}C_{qky} \;-\; \sum_y p^2 C_{py}C_{qy} \\ &= -(p^2+q^2)A_{pq} + \sum_{k,y}\left(\frac{k}{y}\right)^2 C_{pky}C_{qky} \;\equiv\; \frac{1}{4\pi^2}B_{pq}\,. \end{aligned} \tag{6.10}$$

[4]In the concluding section, we will describe another choice of basis, in which the inner product again becomes diagonal.

Let us go over the individual steps of the above calculation. In the first line, we used that $[[Q_0^>, \Sigma], \Sigma] = 0$ and inserted a sum over a complete set of intermediate two-string states. Next we evaluated the matrix elements, using the previously obtained order g_2 results (6.6) and (6.7). Finally, we used the identities (4.12) and [6]

$$\sum_{k,y} \left(\frac{k}{y}\right) C_{p\,ky} C_{qky} = (p+q) A_{p\,q}\,, \tag{6.11}$$

a relation that follows from the fact that the second line in (6.10) must be a symmetric function of p and q (since Z is a symmetric expression). The very last relation in (6.10) is the "unitarity check" of [8, 6] (with the correct sign!). Combined, the answers (6.9) and (6.10), when put back into (6.8), *exactly* reproduce the gauge theory results of [5] and [6]. The X and Y contributions represent the diagrams with nearest and semi-nearest neighbor contractions, and the Z term comprises all non-nearest neighbor interactions.

The knowledge of the second order matrix elements of H and first order mixing terms is sufficient to find the order g^2 corrections to the energy eigenvalues of two-impurity states. In the gauge theory, these correspond to the leading $1/N$ correction to their conformal dimensions.

7. Conclusion

We have presented an interacting string bit model and verified, via a number of non-trivial tests, that it reproduces the non-planar corrections to the gauge theory amplitudes. In particular, for the special class of two-impurity operators, it leads to the same $1/N$ corrections in the conformal dimensions as reported in [5]–[7].

Once this relation between the gauge theory and bit string theory is established, however, we still have the freedom to choose a different basis of states, that is, a different identification between single and multi-string states and single and multi-trace operators in the gauge theory. Given the form (4.1) of the inner product, with S as in (4.7), it is natural to define a new basis of states via

$$|\psi\rangle \to |\widetilde{\psi}\rangle = e^{-g_2 \Sigma/2} |\psi\rangle\,. \tag{7.1}$$

Relative to this new basis of states, the inner product reduces to the standard diagonal one, without any mixing terms between single and multi-string states. The interacting supercharges in this basis can be written as

$$Q = Q_0 + \frac{g_2}{2}[Q_0^< - Q_0^>, \Sigma]\,. \tag{7.2}$$

The strength of string interactions in this basis is in fact $g_2\sqrt{\lambda'}$, rather than g_2: the string theory at $\lambda' = 0$ is free. Note that the new interacting supercharge (7.2) still has only a cubic interaction term; it differs from the original Ansatz in [1] via a (crucial) relative minus sign between $Q_0^<$ and $Q_0^>$, ensuring that (7.2) is hermitean.

We have shown that the superalgebra generated by these charges closes at the linearized level in the fermions. There are two ways in which one can try to improve our expression so that it generates a closed algebra to all orders. One approach would be to try to modify the string bit theory by taking into account that, in the gauge theory, the fermionic impurities carry half a unit of R-charge, and in effect create or destroy half a string bit. This modification may make it possible to write exact supercharges, that are still linear in the fermions.

Alternatively, it should be possible to write non-linear fermionic corrections to Q, designed to produce a closed algebra. Presumably, this leads to an expression for the interaction vertex similar to that of continuum light-cone string field theory [16]–[18], or its SO(4) $\times$ SO(4) invariant modification proposed in [19]. In any case, the string bit theory appears to provide a natural setting for resolving some of the apparent discrepancies between the perturbative gauge theory amplitudes and the small λ' limit of string field theory amplitudes, which are most likely due to an unallowed interchange of limits [20]: to get the continuum string theory one should take the large J limit with *finite* λ', rather than expand around $\lambda' = 0$ and then take the large J limit. We intend to return to these open question in a future work.

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A. Derivation of eq. (6.7)

In this appendix we derive eq. (6.7). Let $\gamma_1 = (1\,2\ldots J)$ denote the single string sector and $\gamma_2 = (J_1 \ldots J_1 - 1)(J_1 \ldots J-1)$ denote the double string sector such that $\gamma_1 \circ \Sigma_{J_1 J} \circ \gamma_2 = 1$. The supersymmetry generators act on these sectors as (here we identify the last site $m = J$ with the 0-th site $m = 0$)

$$Q^{<}_{\gamma_1} = \lambda \sum_{m=0}^{J-1} \left[\left(a^{i\,\dagger}_{m+1} + a^i_{m+1} \right) - \left(a^{i\,\dagger}_m + a^i_m \right) \right] \gamma^i \beta^\dagger_m \,,$$

$$\begin{aligned} Q^{>}_{\gamma_2} = \lambda \sum_{m=0}^{J-1} \beta_m \gamma^i \Big[\left(a^{i\,\dagger}_{m+1} + a^i_{m+1} \right) - \left(a^{i\,\dagger}_m + a^i_m \right) \Big] + \\ + (\beta_{J_1-1} - \beta_{J-1})\, \gamma^i \left[\left(a^{i\,\dagger}_J + a^i_J \right) - \left(a^{i\,\dagger}_{J_1} + a^i_{J_1} \right) \right] . \end{aligned}$$

It is convenient to further decompose $Q^{<}_{\gamma_1}$ into two contributions: a term $Q^{\leq}_{\gamma_1}$ containing all terms of the form $a_i \gamma^i \beta^\dagger$, and a second piece $Q^{\ll}_{\gamma_1}$ consisting of all contributions of the form $a^\dagger_i \gamma^i \beta^\dagger$. Similarly, we define $Q^{\geq}_{\gamma_2}$ and $Q^{\gg}_{\gamma_2}$. We first consider the matrix element

$\langle 2|Q^{\geq}\,\Sigma\,Q^{\leq}|1\rangle$. We compute

$$Q^{\leq}_{\gamma_1} O^p_J = -\frac{\lambda}{J}\sum_{r,l=1}^{J} e^{\frac{2\pi i p(l-r)}{J}} b_l^\dagger(\beta^\dagger_{r-1}-\beta^\dagger_r)$$
$$O^k_{J_1} Q^{\geq}_{\gamma_2} = -\frac{\lambda}{J_1}\sum_{l'=0}^{J_1-1}\sum_{r'=1}^{J_1}\left(b_{l'}e^{-\frac{2\pi i k l'}{J_1}}\right)(\beta_{r'-1}-\beta_{r'})e^{\frac{2\pi i k r'}{J_1}}\,.$$

Taking the inner product yields the sum

$$\frac{\lambda^2}{JJ_1}\left(\sum_{l=0}^{J_1-1} e^{2\pi i\,(\frac{p}{J}-\frac{k}{J_1})l}\right)\times$$
$$\times\left(\sum_{r,r'=1}^{J_1}\delta_{r,r'}+\sum_{r,r'=0}^{J_1-1}\delta_{r,r'}-\sum_{r'=0}^{J_1-1}\sum_{r=1}^{J_1}\delta_{r',r-1}-\sum_{r=0}^{J_1-1}\sum_{r'=1}^{J_1}\delta_{r,r'-1}\right)e^{2\pi i\,(\frac{k\,r'}{J_1}-\frac{p\,r}{J})}\,.$$

Performing the sum and using the operator state correspondence (3.4)–(3.6), while keeping track of the action of the centralizers, gives

$$\langle 2,k,y|Q^{\geq}\,\Sigma\,Q^{\leq}|1,p\rangle = \lambda^2\,\langle 2,k,y|\Sigma|1,p\rangle\left(1-e^{\frac{2\pi i k}{J_1}}\right)\left(1-e^{-\frac{2\pi i p}{J}}\right)$$
$$\langle 2,k,y|\Sigma|1,p\rangle = \frac{1}{J^2}\sqrt{\frac{J-J_1}{JJ_1}}\sum_{l,r=0}^{J_1-1} e^{\frac{2\pi i(py-k)(l-r)}{J_1}} = C_{pky}. \tag{A.1}$$

The other matrix element $\langle 2,k,y|Q^{\gg}\,\Sigma Q^{\ll}|1,p\rangle$ turns out to be equal, at large J, to the one we just computed, up to a "normal ordering" term

$$\begin{aligned}\langle 2,k,y|\bar{Q}^{\gg}\,\Sigma Q^{\ll}|1,p\rangle &= \frac{\lambda^2}{J^2}\sqrt{\frac{J-J_1}{JJ_1}}\left(\sum_{l=0}^{J_1-1} e^{-\frac{2\pi i(k-py)l}{J_1}}\right)\langle\gamma_2|\left(\sum_{r'=0}^{J_1-1} a^i_{r'}e^{\frac{2\pi i k r'}{J_1}}\right)\times\\ &\times\left[\sum_{m=0}^{J-1}\left(a^i_{m+1}-a^i_m\right)\left(a^{i\,\dagger}_{m+1}-a^{i\,\dagger}_m\right)\right]\left(\sum_{r=1}^{J} a^{i\,\dagger}_r e^{-\frac{2\pi i p y r}{J_1}}\right)|\gamma_1\rangle\\ &= 2\lambda^2 J\langle 2,k,y|\Sigma|1,p\rangle\;+\;\langle 2,k,y|Q^{\geq}\,\Sigma\,Q^{\leq}|1,p\rangle\,.\end{aligned} \tag{A.2}$$

The first term arises due to normal ordering and therefore corresponds to a "vacuum fluctuation". In the end, this contribution gets cancelled once we take into account the infinitesimal readjustment of the vacuum induced by the presence of the hopping terms in the hamiltonian. After adding both contributions (A.1) and (A.2), and taking the limit $\frac{k}{J_1}$, $\frac{p}{J}\ll 1$, we arrive the announced result (6.7).

B. Two-loop check of inner product

In this appendix we would like to present an explicit evaluation of the action of the inner product S on the string vacuum

$$\begin{aligned}\langle\gamma_1|S|\gamma_1\rangle &= \langle\gamma_1|e^{\frac{\Sigma_2}{N}}|\gamma_1\rangle\\ &= \langle\gamma_1|\left(1+\sum_{h=1}^{\infty}\frac{1}{(2h)!N^{2h}}\Sigma^{2h}\right)|\gamma_1\rangle\end{aligned} \tag{B.1}$$

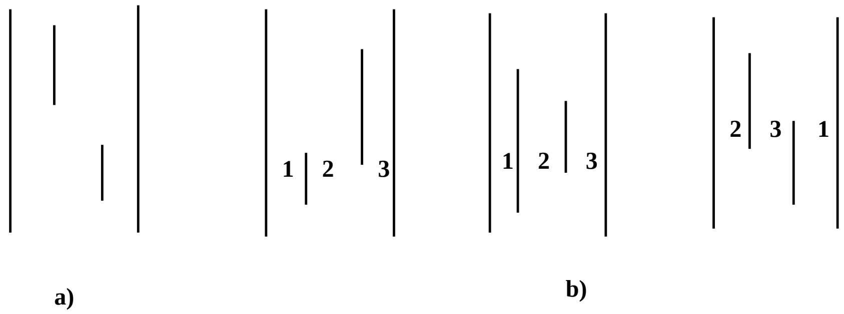

Figure 1: The light-cone string diagram of the $1 \to 2 \to 1 \to 2 \to 1$ process and *b*) the $1 \to 2 \to 3 \to 2 \to 1$ process.

and show that it indeed mirrors a gauge theory calculation

$$\langle \mathrm{Tr}(Z^J)(0)\, \mathrm{Tr}(Z^J)(x) \rangle = \left[1 + \sum_{h=1}^{\infty} \frac{1}{(2h+1)!} \left(\frac{g_2}{2} \right)^{2h} \right] \frac{1}{(4\pi^2 x^2)^J} \tag{B.2}$$

to order ${g_2}^3$. Similar but more involved calculations are needed for higher genera. Alternatively, a more general proof was given in the main body of the paper.

To begin, let us evaluate the genus one contribution to the S-norm of the vacuum. Given a certain long string permutation $\gamma_1 = (12 \ldots J)$ the action of Σ_{ij} splits the string into a two-string state $\gamma_2 = (j\ j+1\ i+2 \ldots i-1)(12 \ldots j-1\ i\ i+1 \ldots J)$. A further permutation Σ_{kl} is needed to rejoin the strings, with $j < k < i-1$ and $1 < l < j, i < l < J$. Thus from

$$\begin{aligned} 1 \to 2 \to 1: \qquad \frac{1}{2!N} \langle \gamma_1 | {\Sigma_2}^2 | \gamma_1 \rangle &= \frac{1}{2!N} \sum_{i=1}^{J} \sum_{j=1}^{i} \sum_{k=i}^{j} \left(\sum_{l=1}^{j} + \sum_{l=i}^{J} \right) 1 \\ &= \frac{1}{2!N} \sum_{i=1}^{J} \sum_{j=1}^{i} (i-j)(J-i+j) = \frac{1}{3!} \left(\frac{J^2}{2N} \right)^2 \end{aligned} \tag{B.3}$$

one derives that the genus one contribution matches the genus one gauge theory normalization of the zero impurities BMN operators.

The genus two calculation follows the same pattern: one first acts with Σ_{ij} to split the string. Next we can rejoin it with Σ_{kl}, split it and join it once more. The net effect will be the square of the previous splitting and joining computation :

$$1 \to 2 \to 1 \to 2 \to 1: \qquad J^8 \frac{1}{3^2 4^2}\,. \tag{B.4}$$

Or, we can decide to further split the string after acting with Σ_{ij}, and join the strings afterwards: $1 \to 2 \to 3 \to 2 \to 1$. Depending on the position of the k, l indices with respect to i, j we distinguish four cases: i) $j < l < k < i$, ii) $l < k < j < i$, iii) $l < j < i < k$, and iv) $j < i < l < k$. In each case the three strings obtained after the action of $\Sigma_{kl}\Sigma_{ij}$

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can be joined in three different ways by acting further with two consecutive transpositions. Implicitly performing the summation over the position of the indices of the latter two transpositions one obtains

$$1 \to 2 \to 3 \to 2 \to 1: \quad i) + ii) + iii) + iv) = \frac{1}{630}J^8 + \frac{1}{840}J^8 + \frac{1}{630}J^8 + \frac{1}{840}J^8 = \frac{1}{180}J^8 \tag{B.5}$$

$$\begin{aligned}
i) &= \sum_{i=1}^{J}\sum_{j=1}^{i}\sum_{k=j}^{i}\sum_{l=j}^{k}\Big[(l-j+i-k)(k-l)(i-j)(j+J-i)+(k-l)(j+J-i)(k-l+j+J-i)\times \\
&\qquad \times (l-j+i-k)+(l-j+i-k)(j+J-i)(l+J-k)(k-l)\Big] \\
ii) &= \sum_{i=1}^{J}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}\Big[(i-j)(k-l)(i-j+k-l)(l+j-k+J-i) + (i-j)(l+j-k+J-i)\times \\
&\qquad \times (l+J-k)(k-l)+(k-l)(l+j-k+J-i)(j+J-i)(i-j)\Big] \\
iii) &= \sum_{i=1}^{J}\sum_{j=1}^{i}\sum_{k=i}^{J}\sum_{l=1}^{j}\Big[(i-j)(k-i+j-l)(k-l)(l+J-k)+(i-j)(l+J-k)(l+J-k+i-j)\times \\
&\qquad \times (j-l+k-i)+(j-l+k-i)(l+J-k)(j+J-i)(i-j)\Big] \\
iv) &= \sum_{i=1}^{J}\sum_{j=1}^{i}\sum_{k=i}^{J}\sum_{l=i}^{k}\Big[(i-j)(k-l)(i-j+k-l)(j+l-i+J-k) + (i-j)(j+l-i+J-k)\times \\
&\qquad \times (l+J-k)(k-l)+(k-l)(j+l-i+J-k)(j+J-i)(i-j)\Big]
\end{aligned} \tag{B.6}$$

Putting everything (B.4), (B.5) together, the genus two contribution to the vacuum S-norm is

$$\frac{1}{4!N^2}\langle\gamma_1|\Sigma_2^4|\gamma_1\rangle = \frac{J^8}{4!N^2}\left(\frac{1}{180}+\frac{1}{3^2 4^2}\right) = \frac{1}{5!}\left(\frac{J^2}{2N}\right)^4 \tag{B.7}$$

once more in perfect agreement with the gauge theory calculation.

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